https://www.linkedin.com/feed/update/urn:li:activity:6497374311073091584

Let $S = \lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \lfloor\sqrt{3}\rfloor + \ldots + \lfloor\sqrt{2019}\rfloor$. Evaluate $\lfloor\sqrt{S}\rfloor$.

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For any $n \in \mathbb{N}$ let $S_n := \sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor$ and $\langle n \rangle := n - \lfloor \sqrt{n} \rfloor^2 \in \left\{0, 1, .., \lfloor \sqrt{n} \rfloor^2 + 2\lfloor \sqrt{n} \rfloor\right\}$.

That is any $n \in \mathbb{N}$ can be uniquely represented in the form $n = m^2 + p$, where $m \in \mathbb{N}$ and $p \in \{0, 1, .., 2m\}$. Then $S_n = \sum_{k=1}^{m^2-1} \lfloor \sqrt{k} \rfloor + \sum_{k=0}^{p} \lfloor \sqrt{m^2 + k} \rfloor = \sum_{k=1}^{m^2-1} \lfloor \sqrt{k} \rfloor + m(p+1)$

and $\sum_{k=1}^{m^2-1} \lfloor \sqrt{k} \rfloor = \sum_{k=1}^{m-1} \sum_{i=0}^{2k} \lfloor \sqrt{k^2 + i} \rfloor = \sum_{k=1}^{m-1} k \sum_{i=0}^{2k} 1 = \sum_{k=1}^{m-1} k(2k+1) = \dfrac{(m-1)m(4m+1)}{6}$

Hence, $S_n = \dfrac{m(4m+1)(m-1)}{6} + m(p+1)$.

For $n = 2019$ we have $m = \lfloor \sqrt{2019} \rfloor = 44, p = 2019 - 44^2 = 83$ and, therefore,

$S = S_{2019} = \dfrac{44 \cdot (4 \cdot 44 + 1)(44 - 1)}{6} + 44 \cdot 84 = 59510$

Hence, $\lfloor \sqrt{S} \rfloor = \lfloor \sqrt{59510} \rfloor = 243$.

Remark.

Since $p = n - m^2$ and $m = \lfloor \sqrt{n} \rfloor$ then $S_n = \dfrac{m(4m+1)(m-1)}{6} + m(n - m^2 + 1) =$

$\dfrac{1}{6} m(6n - 2m^2 - 3m + 5) = \dfrac{m(6n - (2m+5)(m-1))}{6} =$

$\lfloor \sqrt{n} \rfloor \left(n - \dfrac{(2\lfloor \sqrt{n} \rfloor + 5)(\lfloor \sqrt{n} \rfloor - 1)}{6} \right)$.

Thus, $\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor = \lfloor \sqrt{n} \rfloor \left(n - \dfrac{(2\lfloor \sqrt{n} \rfloor + 5)(\lfloor \sqrt{n} \rfloor - 1)}{6} \right)$.